Mechanics of orthogonally reinforced sand

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ABSTRACT- *The response of a cohesionless granular medium reinforced with a uniformly spaced orthogonal network of reinforcement to the action of external stresses is studied. The medium is treated as a continuum. The sand phase is assumed to behave according to the constitutive equation proposed by Poorooshasb and his colleagues and the reinforcement phase according to a set of rules that are described fully in the paper. The constitutive equation derived for the two phase soil (reinforced soil) is objective and hence may be used directly in any analysis involving such media.*

INTRODUCTION

The material reported in this paper describes some preliminary results of joint research conducted in Kyoto and Concordia Universities regarding reinforced sand. The sand phase is assumed to behave as an elasto strain hardening plastic continuum obeying the constitutive relation proposed by Poorooshasb et al (1966,67) modified later by Poorooshasb and Pietruszazck (1986). The reinforcement phase is also treated as a continuum capable of withstanding large tensile but limited shearing stresses for bounded values of deformations. The final constitutive equation for the composite medium is derived by harmonising the deformation of the two phases of the medium and equating the sum of internal forces in the two phases to the external forces acting on the element. The relation obtained between stress and strain rates are objective with a non singular matrix relating the stress and strain rate spaces.

Rectangular Cartesian tensors and small deformation theory is used throughout.

STRESS DEFORMATION PROPERTIES OF THE REINFORCEMENT PHASE

The reinforcement is assumed to be orthogonal and to consist of a series of units placed at equal distances and mutually normal to each other. Fig.(1) shows two such cases.

Although the three dimensional reinforcement, Fig.(1,a), is unlikely to be of great practical interest the analysis presented here will include its treatment for the sake of completeness. The reinforcement scheme shown in Fig. (1,b) is in comon use and is sometimes referred to as sheet reinforcement.

Let the spacial axis of reference be denoted by $x_i (= x_1, x_2, x_3)$ such that they coincide with the directions of the reinforcements. Let $n_i = (n_1, n_2, n_3)$ be the fraction area of the reinforcements in the x_i directions. Then if the stress and strain tensors in the reinforcements are denoted by r_{ij} and
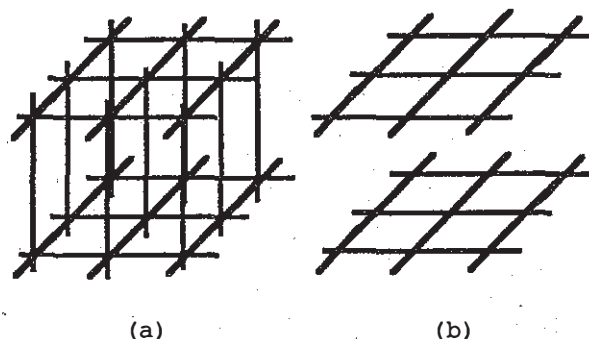

(a)   (b)

Fig.(1)-Orthogonal reinforcement (a) Three dimensional and (b) Two dimensional reinforcements.

ε_{ij} respectively and remembering that the reinforcement phase is being treated as a continuum then the following set of relations exist

$$
\begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{22} \\ r_{23} \\ r_{33} \end{bmatrix} = \begin{bmatrix} En_1 & & & & & \\ & g_{12} & & & 0 & \\ & & g_{13} & & & \\ & & & En_2 & & \\ & 0 & & & g_{23} & \\ & & & & & En_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{bmatrix}
$$

Eq. (1)

where E is Youngs modulus of the material of the reinforcement and g_{12}, g_{13} and g_{23} are equivalent shear moduli of the frame. They are not equal to the shear modulus of the reinforcing material rather to reinforcing structure. For example if the reinforcing sheet, Fig.(2),has a pitch of L (i.e. the distance between adjacent members is L) and the breadth (width) of a typical member is b then using elementary theory of structures the equivalent shear modulus is calculated to be $g=E(b/2L)^3$. Thus for a b/L ratio of 10 say,the value of g is evaluated to be E/8000! The above relation is with reference to a special frame of reference (i.e.when the x_i axis are directed along the reinforcement directions.) To obtain an objective constitutive relation they must be expressed in an arbitrarily chosen frame.To this end note that since both r_{ij} and ε_{ij} are treated as second order tensors then the coefficients relating them to each other must be a fourth order tensor which shall be denoted by C_{ijkl}.Stated
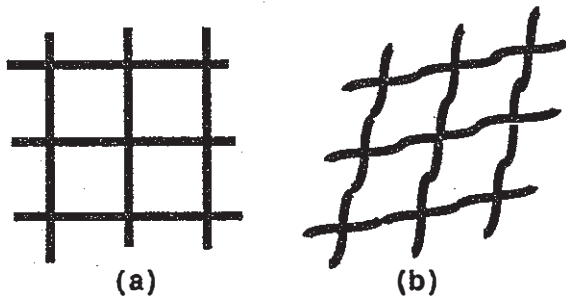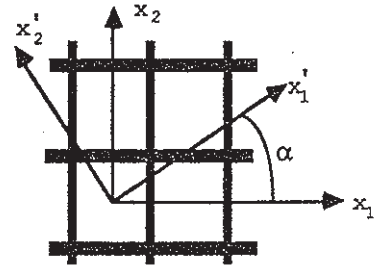

Fig.(3) Key figure.

otherwise C_{ijkl} is a fourth order tensor with principal values given in Eq.(1). The evaluation of the components of C_{ijkl} is straight forward and for the simpler two dimensional case is carried out in detail below. Referring to Fig.(3) let x'_i (i=1,2,3) be a new system of coordinates.It is required to obtain the components of the tensor C in the new system x'_i. The fourth order tensor C has $16(=2^4)$ components in two dimensions but only three constants would appear in the constitutive matrix.When the axis of reference are codirectional with the axis of reinforcements these are;

$$C_{1111}=En_1$$
$$C_{1212}=C_{2121}=g$$
$$C_{2222}=En_2$$

all other components being zero.In the x'_i system of reference and after symmetrizing the matrix the coeficients are obtained as;

$$C'_{1111}=\kappa_1+\kappa_2\cos2\alpha-\kappa_3\sin^2 2\alpha/2$$
$$C'_{1112}=C'_{1211}=-(2\kappa_2\sin2\alpha+\kappa_3\sin4\alpha)/4$$
$$C'_{1122}=C'_{2211}=\kappa_3\sin^2 2\alpha/2$$
$$C'_{1212}=\kappa_2-\kappa_3\cos^2 2\alpha$$
$$C'_{1222}=C'_{2212}=-(2\kappa_2\sin2\alpha-\kappa_3\sin4\alpha)/4$$
$$C'_{2222}=\kappa_1-\kappa_2\cos2\alpha-\kappa_3\sin^2 2\alpha/2$$

where $\kappa_1=(En_1+En_2)/2$, $\kappa_2=(En_1-En_2)/2$, $\kappa_3=(En_1+En_2)/2-g$ and

$$
\begin{bmatrix} r'_{11} \\ r'_{12} \\ r'_{22} \end{bmatrix} = \begin{bmatrix} C'_{1111} & C'_{1112} & C'_{1122} \\ C'_{1211} & C'_{1212} & C'_{1222} \\ C'_{2211} & C'_{2212} & C'_{2222} \end{bmatrix} \begin{bmatrix} \varepsilon'_{11} \\ \varepsilon'_{12} \\ \varepsilon'_{22} \end{bmatrix}
$$

Note that if $n_1=n_2$ then $\kappa_2=0$,

Fig.(2) Rigid grid (a) before shear (b) after shear.

$C'_{1111}+C'_{1122}= \kappa_1$, $C'_{2211}+C'_{2222}= \kappa_1$ and $C'_{1211}+C'_{1222}=0$. Thus for a state of pure compression whereby $\varepsilon'_{11}= \varepsilon'_{22}=\varepsilon$, $\varepsilon'_{12}=0$ the above relations yield $r'_{11}=r'_{22}=\kappa_1\varepsilon$, $r'_{12}=0$ as expected.

Finally since the deformation response of the reinforcing grid to stress is assumed to remain linear during the loading process (i.e. E and g are assumed to remain constant) then it is rational to state

$$\dot{r}_{ij}=C_{ijkl}\dot{\varepsilon}_{kl} \qquad (2)$$

where $\dot{r}_{ij}$ and $\dot{\varepsilon}_{kl}$ are the stress rate and the strain rate tensors respectively.

Having obtained the coefficients of the deformation matrix for the reinforcement the constitutive relation of the sand phase is examined next.

STRESS DEFORMATION PROPERTIES OF THE SAND PHASE

The constitutive relation proposed for sand assumes the existence of a global plastic potential ϕ and a local potential ϕ' which is derived from the global plastic potential. During virgin loading the plastic strain rate tensors are derived from ϕ and during stress reversals from ϕ'. Thus the strain rate tensor is related to the stress rate tensor by the relation;

$$\dot{\varepsilon}_{ij}=\dot{\varepsilon}_{ij}^{elastic} + \dot{\varepsilon}_{ij}^{plastic}$$
$$= E_{ijkl}\dot{s}_{kl} + \dot{\lambda}\frac{\partial\phi}{\partial s_{ij}} \qquad (3)$$

where s_{ij} is the effective stress tensor associated with the sand phase, E_{ijkl} are the elastic moduli and λ' is the loading index. Its magnitude depends on whether the sand is loading, unloading (in which case its magnitude is zero) or reloading (in which case its magnitude is related to a conjugate quantity associated with the bounding surface, Dafalias(1982)). Specifically the loading index is given by the relation

$$\dot{\lambda} =h\frac{\partial f}{\partial s_{ij}}\dot{s}_{ij} \qquad (4)$$

where f is the yield function and h is a parameter that determines the magnitude of plastic strain increment tensor. It is related to the history of loading of the element.

Denoting by I_1, J_2 and J_3 the first invariant of the stress tensor and the second and third invariants of the stress deviation tensor respectively (i.e. $I_1=s_{11}$, $J_2=\sqrt{S_{ij}S_{ij}}$ and $J_3= S_{ij}S_{jk}S_{ki}$, S_{ij} being equal to $s_{ij}- I_1\delta_{ij}/3$) the yield function F may be expressed through the relation

$$F= J_2-\eta(\varepsilon)I_1 g(\theta)=0$$

during the virgin loading process. In this last equation $\eta(\varepsilon)$ records the history of loading in terms of the total plastic strain ε, and θ is a function of the first and the third invariants (it is equivalent to Lode's angle). The function $g(\theta)$ has a certain symmetrical form about $\theta=n\pi/3$ for an isotropic sand. If $g(\theta)$=constant then the extended von Mises yield surface would obtain.

It must be emphasized that only during virgin loading the yield function f coincides with the bounding surface F. In general (e.g. during stress reversals) such a relation does not exist and indeed the kinematics of the yield surface within the bounding surface follows certain rules which can not be presented here. The interested reader may refer to the papers by the second author and his colleagues on stress deformation of sand.

STRESS DEFORMATION PROPERTIES OF THE REINFORCED MEDIUM.

Combination of Eqs.(2) and (3) results in the equation

$$\dot{r}_{ij}= C_{ijkl}(E_{klpq}\dot{s}_{pq}+\dot{\lambda}\frac{\partial\phi}{\partial s_{kl}})$$

which upon substitution from Eq.(4) for λ reduces to

$$\dot{r}_{ij}= C_{ijkl}(E_{klpq}+h\frac{\partial\phi}{\partial s_{kl}}\frac{\partial f}{\partial s_{pq}})\dot{s}_{pq} \qquad (5)$$

Equation (5) is the first equation relating the stress tensor in the reinforcement r_{ij} to stress tensor in the sand phase s_{ij}. In its derivation it has been tacitly assumed that the sand and the reinforcing grid deform harmoniously i.e. no slippage take place between the two phases of the composite medium.

A second equation to relate the stress tensors r_{ij} and s_{ij} may be obtained noting that their sum must equal the
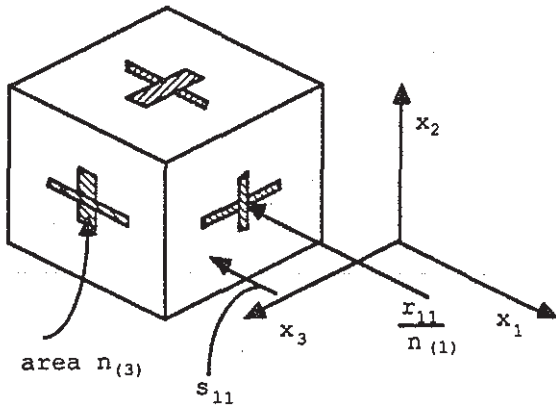
Fig.(5) **Schematic representation** of unit volume of reinforced sand

external applied stress σ_{ij}. Before this is done however it is worth stating that the r_{ij} tensor is a tensor of apparent stresses; see Fig.(5). The actual stress acting in the reiforcing bars is $r_{ij}/n_{(i)}$ where $n_{(i)}$ is the area fraction of the reinforcement in the ith face of the control volume of the medium. The situation is quite similar to flow of fluid through porous media where the apparent velocity is used in the constitutive equation of D'Arcy and is related to the loss of energy. The relation between apparent velocity and the actual velocity of flow is precisely the relation stated above i.e. $V_i = v_i/n_{(i)}$.

Thus the relation
$$r_{ij} + s_{ij}(1 - n_{(i)}) = \sigma_{ij} \qquad (6)$$
exists. Now if $n_{(i)}$ is small compared to unity then
$$r_{ij} + s_{ij} = \sigma_{ij} \qquad (6,a)$$
and a similar expression may be written for the rates of r_{ij}, s_{ij} and σ_{ij}.

Let for reason of convenience;

$$D_{ijpq} = C_{ijkl}\{E_{klpq} + h(\partial\phi/\partial s_{kl})(\partial f/\partial s_{pq})\}$$

Then Eq.(5) may be written as

$$\dot{r}_{ij} = D_{ijpq}\dot{s}_{pq} \qquad (7)$$

and when combined with Eq.(6) yields

$$\dot{s}_{ij} + D_{ijpq}\dot{s}_{pq} = \dot{\sigma}_{ij} \qquad (8)$$

Equation (8) relates the stress in the sand phase to the externally imposed stresses.

But $s'_{ij} = s_{pq}\delta_{ip}\delta_{jq}$ where δ_{ij} is the Kronecker's delta. Thus Eq.(8) may be restated as

$$(\delta_{ik}\delta_{jl} + D_{ijkl})\dot{s}_{kl} = \dot{\sigma}_{ij} \qquad (9)$$

Before proceeding further it is worth noting that the fourth order tensor D_{ijkl} is, in all likelihood, a singular tensor (i.e. the inverse of the associated matrix may not exist). Such singularities may be the results of the reinforcement constitutive matrix (e.g. when no reinforcement is present in one of the directions x_i) or could be introduced if the sand is assumed to be a rigid plastic (rather than an elastic plastic) material.

The tensor $K_{ijkl} = D_{ijkl} + \delta_{ik}\delta_{jl}$ is, however, a non singular tensor. It is also an antisymmetric tensor since it is the sum of a symmetric tensor (the unit tensor $\delta_{ik}\delta_{jl}$) and the product of a symmetric (the C_{ijkl}) tensor and an antisymmetric tensor (the tensor of the elasto-plastic coefficients in the constitutive relation for sand).

Since K_{ijkl} is deemed to be non singular then Eq.(9) may be rewritten as

$$\{\dot{s}\} = [K]^{-1}\{\dot{\sigma}\} \qquad (10)$$

But from Eqs.(3) and (4)

$$\{\dot{\varepsilon}\} = [H]\{\dot{s}\}$$

where $[H] = E_{ijkl} + h(\partial\phi/\partial s_{ij})(\partial f/\partial s_{kl})$. Thus,

$$\{\dot{\varepsilon}\} = [H][K]^{-1}\{\dot{\sigma}\} \qquad (11)$$

which is the desired relationship between the stress and the strain tensor for the reinforced sand medium.

The various operations outlined above will be demonstrated by means of a simple example in the next section.

AXIAL LOADING OF AN ELEMENT REINFORCED IN A DIRECTION NORMAL TO THE AXIS OF THE MAJOR PRINCIPAL STRESS

Consider a sample of reinforced sand subjected to a state of axial loading whereby $\sigma_2 = \sigma_3$ remains constant with σ_1 increasing (triaxial compression test). The reinforcement is assumed to be of the sheet type with its plane normal to the direction of action of σ_1, Fig.(6).
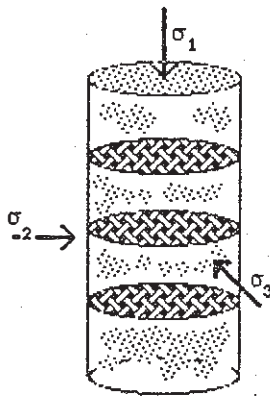
Fig.(6) Transversely reinforced sand in triaxial loading state

Fig.(7)- Yield loci and plastic potential curves for sand in triaxial compression.

Here $n_1 = 0$ and it is assumed that $n_2 = n_3$. This assumption is made only in view of the fact that otherwise it would be necessary to deal with a general state of stress for the sand phase: a discussion which is outside the space limitations of the present paper.

For the reinforcement phase the C matrix assumes the simple form;

$$\{\dot{r}\} = [C]\{\dot{\varepsilon}\}$$

$$[C] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & En & 0 \\ 0 & 0 & En \end{bmatrix}$$

referring to principal directions.

The sand is assumed to be rigid plastic and for convenience the functions f, ϕ and h shall be expressed in terms of σ_1 and $\sigma_3 (= \sigma_2)$. In Fig.(7) are shown the f=const. and ϕ=const. curves associated with compression loading $(\sigma_1 > \sigma_3)$.

If the state of stress experienced by the element is within the zone bounded by th two radial lines marked "critical state line" and the "hydrostatic consolidation line" then the sample contracts upon loading. Furthermore if the state of stress is to the right of the "k_o consolidation line" the strain components $\varepsilon_2 (= \varepsilon_3)$ would be positive (equivalent to a negative Poisson's ratio). This zone (i.e the zone bounded by the radial lines marked k_o and hydrostatic consolidation) is absent in preloaded or compacted samples and presents some peculiar behaviour. For
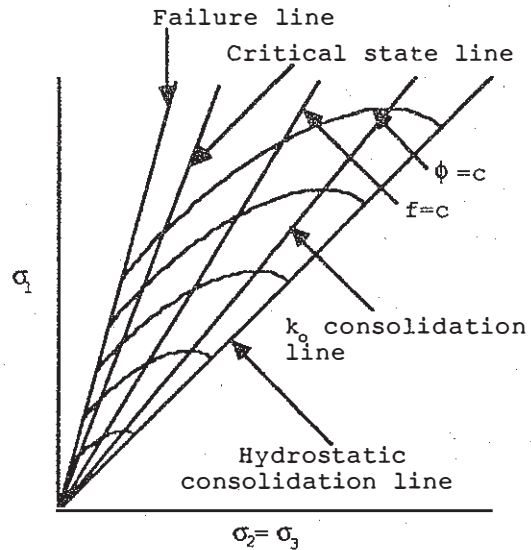
example it may be shown that a solution to certain type of problems may not exist at all when working in this zone of the stress space and assuming rigid plasticity.

If the state of stress is within the zone bounded by the radial lines marked "failure" and "critical state" then a loading of the sample produces expansion.

Now the matrix relating the strain rate tensor to stress rate tensor [[B] of Eq.(11)] for the sand is;

$$h \cdot \begin{bmatrix} f_{,1}\phi_{,1} & f_{,2}\phi_{,2} & f_{,3}\phi_{,3} \\ f_{,1}\phi_{,1} & f_{,2}\phi_{,2} & f_{,3}\phi_{,3} \\ f_{,1}\phi_{,1} & f_{,2}\phi_{,2} & f_{,3}\phi_{,3} \end{bmatrix}$$

Multiplying the above two matrices , adding the unit matrix [1] results the matrix [K] which has an inverse;

$$\frac{1}{det} \begin{bmatrix} 1+ \\ \beta(f_{,2}\phi_{,2}+f_{,3}\phi_{,3}) & 0 & 0 \\ -\beta f_{,1}\phi_{,2} & 1+ \\ \beta f_{,3}\phi_{,3} & -\beta f_{,3}\phi_{,3} \\ -\beta f_{,1}\phi_{,3} & -\beta f_{,2}\phi_{,3} & 1+ \\ \beta f_{,2}\phi_{,2} \end{bmatrix}$$

55

where for convenience $\partial f/\partial\sigma_1$ has been shown by $f_{,1}, \partial\phi/\partial\sigma_1$ by $\phi_{,1}$ and so on, $\det=1+\beta(f_{,2}\phi_{,2}+f_{,3}\phi_{,3})$ and $\beta=$Enh.

In a conventional triaxial test

$$\dot{\sigma}_2=\dot{\sigma}_3=0 \; ; \; f_{,2}=f_{,3} \text{ and } \phi_{,2}=\phi_{,3} \; .$$

Therefore the the principal components of s_{ij} are obtained from the relations

$$\begin{matrix} \dot{s}_{11}=\dot{\sigma}_1 & \text{(a)} \\ \dot{s}_{22}=\dot{s}_{33}=-\dfrac{\beta\phi_{,3}}{\sigma_3\det}\dot{\sigma}_1 & \text{(b)} \end{matrix} \Biggr\} \quad (12)$$

remembering that for the particular type of loading envisaged $f=\sigma_1/\sigma_3$ and hence $\partial f/\partial\sigma_1=f_{,1}=1/\sigma_3$.

From Eq.(12,a) it is seen that the rate of increase of sand stress in the axial direction is equal to the axial stress imposed on the soil. This is indeed as expected since no reinforcements exist in the axial direction.The rate of increase of s_{33} on the other hand is a function of the position of the stress point s_{ij} in the stress space, Fig.(7). If the stress point is in the zone bounded by k_o and hydrostatic lines the component $\phi_{,3}$ is positive and hence from Eq.(12,b) there will be a decrease in the magnitude of s_{33}.This is tantamount to saying that when loading in this zone ,as far as the sand is concerned,a decrease in the confining pressure is experienced. This would lead to larger axial strains that when the soil is tested in the virgin state.This point whilst of some theoretical interest is unlikely to be of great practical value since as mentioned before preloaded or compacted soils do not exhibit such behaviour. Once the state of stress in the sand passes to a position on the left of the k_o line $\phi_{,3}<0$ and hence the rate of change of s_{33} is positive, Eq. (12,b).The soil is progressively getting stiffer (by virtue of the increase in the confining pressure s_{33}) and hence the magnitude of both axial and lateral strains would fall below the values measured for the non reinforced sand.