Simple Analysis of Deformation of Sand-Sausages

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ABSTRACT: The paper deals with simple analysis of deformation of cylindrical shells filled with the material of known unit weight. The aim of this article is to find a shape of such a shell as function of internal pressure and the height of cross - section. The solution is found with the help of numerical methods. The obtained results are of some importance in the analysis of sand-sausages. They were compared with Silvester's ones (1986) and a good agreement was ascertained.

1 INTRODUCTION

Cylindrical shells, filled with sand or mortar (sand-sausages) are used for bank protection either in marine or fluvial structures. Shells filled with water or air, laid across the main stream and fixed to bed are used for forming dams. Proper design of this kind of structures requires a good knowledge of forces in the shell material as well as the shapes of cylinders' cross-section. The aim of this paper is a simple analysis of deformation of the structure that consists of cylindrical shell filled with water.

2 ASSUMPTIONS

The following assumptions are accepted in this paper:
- there is only the membrane state of stress at the covering material,
- plane strain state is considered,
- no concentrated loads act on a structure,
- shell's own weight is neglected,
- the shell is filled with material characterised by the known unit weight,
- there is no friction between the structure and the subsoil.

3 FORMULATION OF THE PROBLEM

The aim of this analysis presented here is to find a shape of shell's cross - section as function of its given height H and given hydrostatic pressure $p_o = h\gamma$ as well as computed membrane force N_ϕ (see Fig.1).

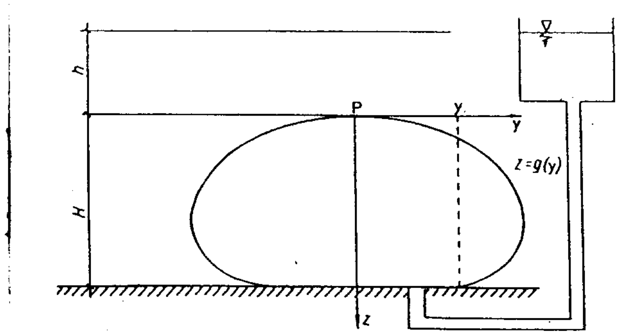


Fig.1 Cylindrical shell's cross-section

Equilibrium equations for cylindrical shell are following:

$$\frac{\partial N_x}{\partial x}\cdot dx\cdot r\cdot d\phi+\frac{\partial N_{\phi x}}{\partial \phi}\cdot d\phi\cdot dx+p_x\cdot dx\cdot r\cdot d\phi=0 \qquad (1)$$

$$\frac{\partial N_\phi}{\partial \phi}\cdot d\phi\cdot dx+\frac{\partial N_{x\phi}}{\partial x}\cdot dx\cdot r\cdot d\phi+p_\phi\cdot dx\cdot r\cdot d\phi=0 \qquad (2)$$

$$N_\phi\cdot dx\cdot d\phi-p_r\cdot dx\cdot r\cdot d\phi=0 \qquad (3)$$

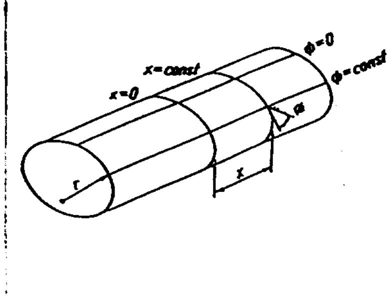


Fig 2 System of co-ordinates

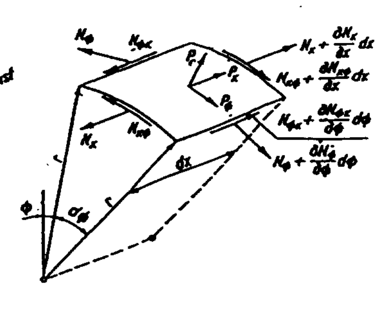


Fig 3. Membrane forces

Because of the accepted assumptions there are:

$p_x = 0; \quad p_\phi = 0; \quad N_x = N_{x\phi} = N_{\phi x} = 0$ (4)

and Eqs.(1) - (3) simplify to the following form:

$$\frac{1}{r}\cdot\frac{\partial N_\phi}{\partial \phi} = 0 \tag{5}$$

$$N_\phi = p_r \cdot r \tag{6}$$

It follows from Eq.(5) that the value of N_ϕ does not depend on ϕ and is constant along circumference of the cross section.

4 SPECIAL CASES

The following three special cases will be considered, depending on the value of internal pressure at point P (see Fig.1):

A) $p_o >> \gamma H$; B) $p_o \neq 0$; C) $p_o = 0$

Because of the symmetry of the problem with respect to the x,z plane we shall consider only one half of the structure cross section, as shown in Fig.4.

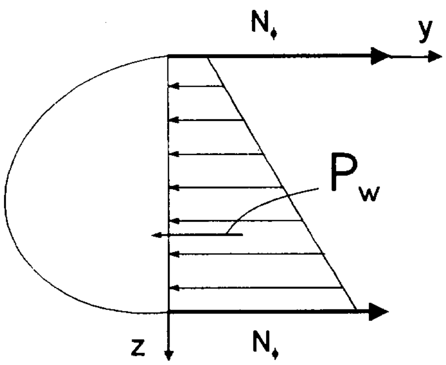
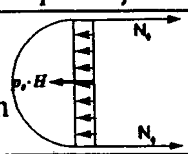
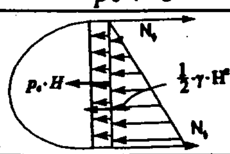
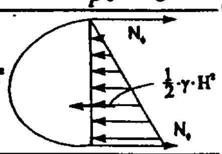


Fig. 4 Considered forces

P_w denotes there the resultant force from the other part of the structure. Table1 shows the basic parameters, i.e. p_r, P_w and N_ϕ, for the above mentioned special cases.

Table 1. Basic parameters for special cases

p_o	$p_o >> \gamma \cdot H$	$p_o \neq 0$	$p_o = 0$
forces diagram			
p_r	p_o	$p_o + \gamma \cdot z$	$\gamma \cdot z$
p_w	$p_o \cdot H$	$0{,}5\gamma H^2 + p_o H$	$0{,}5 \cdot \gamma \cdot H^2$
$N\phi$	$0{,}5 \cdot p_o \cdot H$	$0{,}25\gamma H^2 + 0{,}5 p_o H$	$0{,}25 \cdot \gamma \cdot H^2$

5 METHOD OF CALCULATION

5.1 Case A ($p_o >> \gamma H$)

It follows from Eq.(6):

$$r = N_\phi / p_r \tag{7}$$

Substitution of N_ϕ and p_r, from Table1, into (7) gives:

$$r = \frac{p_o H}{2 p_o} = \frac{H}{2}. \tag{8}$$

As we could expect we have obtained the circular cross section in this special case.

5.2 Cases B($p_o \neq 0$) and C ($p_o = 0$)

Eq.(7) also valid in these cases. Because of variation of p_r along the height of the shell (see Table1), the solution is not as easy as in the previous case.

In order to compute the cross section it is necessary to find formulas for the radius of curvature.

There are two methods of dealing with this problem.

5.2.1 Explicit approach

In differential notation the curvature is given by:

$$k = \frac{1}{r} = \frac{z''}{(1+z'^2)^{3/2}} \tag{9}$$

From Eqs.(9) and (6) one obtains the following differential equation:

$$\frac{d^2 z}{dy^2} = \frac{p_r}{N\phi}\left[1+\left(\frac{dz}{dy}\right)^2\right]^{3/2} \tag{10}$$

The solution of above equation and following boundary conditions: z [0] = 0; dz/dy[0] = 0 defines the cross sectional shape of the shell.

For the extremum point of Eq.(10), the value of dz/dy is unknown, so we must change the procedure and take z as an independent variable. We get the following equation:

$$\frac{d^2 y}{dz^2} = -\frac{p_r}{N\phi}\left[1+\left(\frac{dy}{dz}\right)^2\right]^{3/2} \tag{11}$$

5.2.2 Alternative geometrical approach

An alternative approach to that presented previously is based on geometrical considerations. Let us consider cylindrical shell's meridian presented in Fig.5, r_2 means a distance measured along normal to meridian from his intersection with the axis of rotation to middle plane.

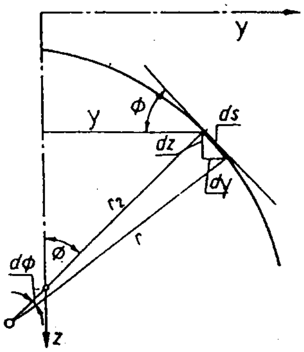


Fig.5 Cylindrical shell's meridian

There is : $ds = r \cdot d\phi \Rightarrow d\phi = \dfrac{ds}{r}$ (12)

where ds denotes the differential length of the shell.

There is also: $dy = ds \cdot \cos\phi$. (13)

Thus: $\dfrac{dy}{d\phi} = r \cdot \cos\phi \Rightarrow \dfrac{1}{r} = \dfrac{\cos\phi \cdot d\phi}{dy} = \dfrac{d\sin\phi}{dy}$. (14)

Substitution of Eq.(14) into Eq.(6) leads to the following equation:

$$\frac{d\sin\phi}{dy} = \frac{p_r}{N\phi} \qquad (15)$$

which together with the geometrical dependence:

$$tg\phi = \frac{dz}{dy} \qquad (16)$$

defines the problem.

Substitution of the identity: $tg\phi = \sin\phi / \sqrt{1-\sin^2\phi}$ and introducing the new variable: $u = \sin\phi$ we get the system of differential equations defining cylindrical shell's cross section in the following form:

$$\frac{du}{dy} = \frac{p_r}{N\phi} \ , \ \frac{dz}{dy} = \frac{u}{\sqrt{1-u^2}} \ . \qquad (17)$$

We use the boundary conditions: u[0]=0: z[0]=0.
The system of equations (17) gives the solution in the following ranges of the angle $\phi : \phi \le 50^0$ and $\phi \ge 140^0$.

For $50^0 < \phi < 140^0$ we must change the procedure and take z as an independent variable (y is not proper independent variable, when ϕ is close to 90^0).
Then the system of equations takes the form:

$$\frac{d\cos\phi}{dz} = \frac{p_r}{N\phi} \ , \ ctg\phi = \frac{dy}{dz} \ . \qquad (18)$$

Introducing the new variable $v = \cos\phi$ and using the identity: $\cos\phi = ctg\phi / \sqrt{1+ctg^2\phi}$ one gets the following system of differential equations:

$$\frac{dv}{dz} = \frac{p_r}{N\phi} \ , \ \frac{dy}{dz} = \frac{v}{\sqrt{1-v^2}} \ . \qquad (19)$$

In order to define the problem we have to add respective boundary conditions. They are corrected with the solution of the problem for $\phi = 50^0$.

5.2.3 Length of contact between subsoil and shell

In the case A here is the point contact between the shell and the subsoil (cf. 5.2.1).
In the cases B and C we have the linear contacts, as shown in Figs.6 and 7. The lengths of these contacts are expressed by the following relations:

$$2 \cdot y'(\gamma \cdot H' + p_o) = Q \Rightarrow y' = \frac{Q}{2(\gamma \cdot H' + p_o)} \qquad (20)$$

$$2 \cdot y \cdot \gamma \cdot H = Q \Rightarrow y = \frac{Q}{2 \cdot \gamma \cdot H} \qquad (21)$$

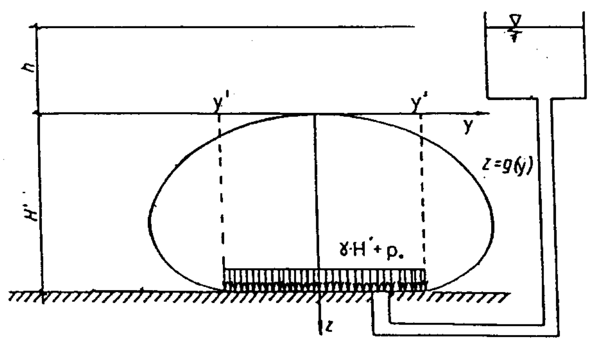


Fig.6 Length of contact between shell and subsoil - case B

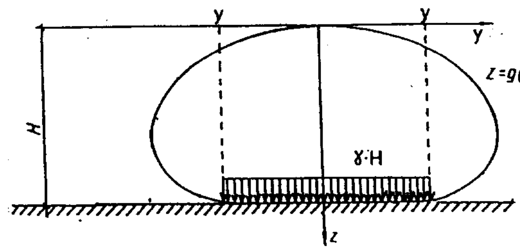


Fig.7 Length of contact between shell and subsoil - case C

6 RESULTS AND CONCLUSIONS

Eqs. (10),(11),(17) and (19) were solved numerically for different initial parameters and the results were compared with experimental data presented in Silvester (1986). Particularly we have compared only the two results: the tensile force N_ϕ in the shell and the length of the contact L' between the structure and the subsoil.
The comparison was made for various heights H of the structure and various values of pressure p_o. The results are shown in Tables 2 and 3. It seems that a good agreement was obtained because the differences between theoretical predictions and Silvester's data do not exceed the value of 3 %.

Table 2 Tensile force in the shell - comparison of computed results with Silvester's experimented data ($\gamma = 1{,}4$ kG/m³)

H[m]	h[m]	N_ϕ[kG]	T_s[kG]	T_s/N_ϕ	T_s/N_ϕ[%]
1,0	1,27	1,239	1,202	0,970	3
0,9	0,64	0,687	0,693	1,009	1
0,8	0,33	0,409	0,397	0,97	3
0,7	0,22	0,279	0,286	1,025	2,5

where:
N_ϕ- force calculated by use of formulas from Table 1
T_s - value of force in the shell presented by Silvester (1986)

Table 3 Length of contact segment between the shell and the subsoil ($\gamma = 1{,}4$ kG/m³)

H[m]	h[m]	Q[kG]	L'[m]	L_s[m]	L'/L_s	L'/L_s[%]
1,0	1,27	1,05γ	0,46	0,46	1,00	0
0,9	0,64	0,99γ	0,64	0,65	0,98	2
0,8	0,33	0,95γ	0,84	0,82	1,02	2
0,7	0,22	0,89γ	0,96	0,94	1,02	2

where:
L' - length of contact segment calculated by use of formulas from point 5.2.3
L_s - length of contact segment presented by Silvester
Q - weight of material filling the shell

The following conclusions can be drawn from the considerations presented:
1. The shape of cylindrical shell's cross section depends on such parameters as: the height of the shell, the value of pressure in the highest point, the own weight, the shell tensile forces, length of circumference of the cross section. These parameters are correlated to each other.
2. The method of defining the cylindrical shell cross section depends on the value of pressure acting inside the shell (cf.cases A, B, C).
3. In the case when the value of the pressure acting in its highest point is much bigger than the value of the hydrostatic pressure, the cross section of the shell takes the shape of the circle.
4. In two extreme cases i.e. when pressure in the highest point of the shell is zero, or when its value is comparable with the value of the hydrostatic pressure, we can find the shape of the cross section using numerical methods. In this article the two types of differential equations were presented (system of linear differential equations and second order differential equation), solutions of which define the shape of the shell cross section. Obtained results by use of both of them are compatible (i.e. the difference is lower than 1%).
5. In the paper several formulas defining length of contact segment between subsoil and shell are presented. Obtained values were compared with Silvester's data and a good agreement was shown. The length of contact between the subsoil and the shell decreases with the increasing values of the pressure at the highest point of the shell.

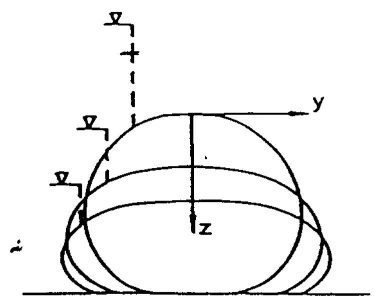


Fig 7 Cylindrical shell's cross-section. Length of circumference = const; H,$p_o \neq$const

6. It is possible to calculate the value of the shell tensile force, which is constant along the circumference, by using equilibrium condition of the shell in y - axis direction. Obtained results are comparable to Silvester ones.

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