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Limit analysis of soil structures subjected to constraints by reinforcement

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ABSTRACT: Many reinforcement methods of soil structures have been proposed and used in engineering practices. This paper aims to present a theoretical methodology to evaluate the safety of soil structures with reinforcement members from the point of limit analysis. The upper bound method is extended to deal with rigid-work hardening materials. Its mathematical properties are also discussed.

1 INTRODUCTION

Many reinforcement methods of soil structures have been presented. They have great advantages in engineering practices. Then, what are key issues of reinforcement mechanisms? This is a simple motivation of this paper.

Stability problems of soil structures with reinforcement members are discussed from the point of limit analysis.

In many engineering practices, the limit equilibrium method is used to estimate the stability of reinforced soil structures. In the previous study, theoretical properties of the limit equilibrium method was discussed and concluded as an approximate method of the upper bound analysis (Kobayashi 2000b). The effect of reinforcement members are usually replaced as member forces in engineering design. Then, according to the limit equilibrium method, equilibrium of forces or moments of the region of interest are calculated to estimated the stability of the system.

However, it must be noticed that member forces are not known prior to the solution. In other words, a collapse mechanism of a system depends on the distributions and the strengths of reinforcement members via deformation restriction effect. Therefore, actual member forces essentially depend on the collapse mechanism (= solution of the analysis). In this sense, a kinematic approach to the rigid plastic boundary value problems — the upper bound method — is one of the promising methods.

As a simple and a typical example, a stability problem of self-standing type sheet pile walls has been investigated and discussed by the author (Kobayashi 2000a).

Interactions between soil structures and reinforcement members must be considered carefully. From the point of limit analysis, deformation mechanisms of soil structures with reinforcement members can be characterised by the following two features;

- Rigidities of geo materials and reinforcement members are very different. Some reinforcement materials are very ductile. Geo-textiles and geomembranes are typical examples. On the other hand, all most all deformation of soil materials can be treated as plastic deformation. Therefore, deformation (or strain) levels of the reinforcement members at their peak strengths and deformation levels of the geo materials at their peak may be mismatched.

- Many geo materials show apparent strain softening behaviours. For example, stress-strain relations of over consolidated clays or dense sands show their peak strengths and decrease the strengths to their residual values with the increase of shear deformation. Then, it must be noted that deformation levels should be kept within a certain magnitude to utilise the potential strength of the geo material.

It is impossible to deal with these two features by the ordinary upper bound analysis. Therefore, an extended method of the upper bound method is proposed and discussed in this paper.

2 PROPOSED METHOD

2.1 *Fundamentals of upper bound method*

Let us consider a stress field $\underset{\sim}{Q}$ satisfying a yield condition, i.e. $f(\underset{\sim}{Q}) = 0$. Let $\dot{\underset{\sim}{q}}^p$ be plastic strain rates corresponding to the stress field $\underset{\sim}{Q}$. Let us also consider another stress field $\underset{\sim}{Q}^*$ which never violates the yield condition at same material points. Hill assumed the following inequality called "principle of maximum plastic work" (Hill 1948).

$$(\underline{Q} - \underline{Q}^*) \cdot \underline{\dot{q}}^p \geq 0 \tag{1}$$

Geometrical interpretation of equation (1) is well known, as

- Yielding surface is convex in stress space.
- Plastic strain rate $\underline{\dot{q}}^p$ is always parallel to the outward normal direction of yielding surface $(\partial f / \partial \underline{Q})$.

Thus, the associated flow rule is derived from Hill's assumption,

$$\underline{\dot{q}}^p = \dot{\lambda} \frac{\partial f}{\partial \underline{Q}}. \tag{2}$$

Equation (1) can be represented in the large as,

$$\int_V (\underline{Q} - \underline{Q}^*) \cdot \underline{\dot{q}}^p dV + \int_\Gamma (\underline{Q}_n - \underline{Q}_n^*) \cdot [\underline{\dot{u}}] dS \geq 0 \tag{3}$$

for a region V and on a velocity discontinuity plane Γ, where $\underline{Q}_n$ and $[\underline{\dot{u}}]$ denote the surface traction and the amount of velocity discontinuity on Γ, respectively.

Let us consider any kinematically and plastically admissible velocity fields $\underline{\dot{u}}$, which satisfy velocity ~ strain rate relation, boundary conditions on the velocity boundaries and the associated flow rule. According to the constitutive relations (2), a corresponding stress field $\underline{Q}^{(k)}$ to the assumed velocity field can be found. It should be noted that stress field $\underline{Q}^{(k)}$ is not always determined uniquely, nor statically admissible. On the other hand, let the exact solution for a stress field be $\underline{Q}^{(t)}$.

Principle of maximum plastic work is applied for this problem;

$$\int_V (\underline{Q}^{(k)} - \underline{Q}^{(t)}) \cdot \underline{\dot{q}}^p dV + \int_\Gamma (\underline{Q}_n^{(k)} - \underline{Q}_n^{(t)}) \cdot [\underline{\dot{u}}] dS \geq 0. \tag{4}$$

After some arrangements, the following inequality can be derived

$$D^{(k)} \geq D^{(t)}, \tag{5}$$

where the superscripts k and t implies the terms of the kinematically admissible field and the exact field, respectively, and

$$D^{(k)} = \int_V \underline{Q}^{(k)} \cdot \underline{\dot{q}}^p dV + \int_\Gamma \underline{Q}_n^{(k)} \cdot [\underline{\dot{u}}] dS \tag{6}$$

$$D^{(t)} = \int_V \underline{Q}^{(t)} \cdot \underline{\dot{q}}^p dV + \int_\Gamma \underline{Q}_n^{(t)} \cdot [\underline{\dot{u}}] dS$$
$$= \int_{S_T} \underline{\bar{T}} \cdot \underline{\dot{u}} dS + \int_{S_D} \underline{T}^{(t)} \cdot \underline{\bar{v}} dS + \int_V \underline{b}^{(t)} \cdot \underline{\dot{u}} dV. \tag{7}$$

This inequality insists that internal dissipation $D^{(k)}$ by a certain trial kinematically admissible velocity field is never less than external plastic work $D^{(t)}$ for the actual limit load.

2.2 *Linear programming formulation of upper bound method*

Linearisation of yielding functions. Linear programming formulation is convenient to the theoretical consideration of the proposed method. By introducing piecewise linearisation of yielding function, limit analysis can be formulated in the form of linear programming (LP). This formulation was firstly introduced by Maier (Maier 1970), and developed in the field of structural plasticity (Maier & Munro 1982; Maier & Lloyd Smith 1986; Lloyd Smith 1990).

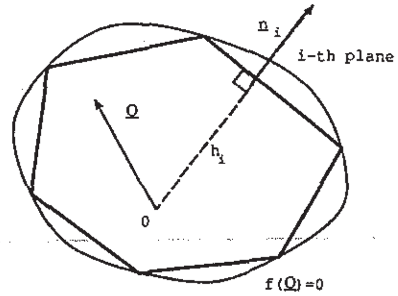

Figure 1. Piecewisely linearised yielding locus.

Yielding function can be expressed as a closed convex locus containing the origin in the stress space, according to Hill's maximum plastic work principle. This locus is then piecewisely linearised by intersectional polyhedron, as shown in figure 1. This piecewisely linearised yielding locus also contains the convexity.

Let $\underline{n}_i$ be the outward normal unit vector of the i-th plane and h_i be the distance from the origin to the i-th plane. Then, a condition that an arbitrary stress state $\underline{Q}$ exists within or on the yielding locus can be expressed as

$$\underline{n}_i \cdot \underline{Q} \leq h_i. \tag{8}$$

Let $\underline{\underline{N}}$ be a tensor arranged each outward unit normal vectors $\underline{n}_i$ and $\underline{h}$ be a vector arranged each distances h_i, a stress state $\underline{Q}$ which is not violating the yielding condition can be expressed as

$$\underline{\underline{N}}^T \underline{Q} \leq \underline{h}, \quad \text{where } \underline{\underline{N}} = (\underline{n}_1, \underline{n}_2, \cdots, \underline{n}_n). \tag{9}$$

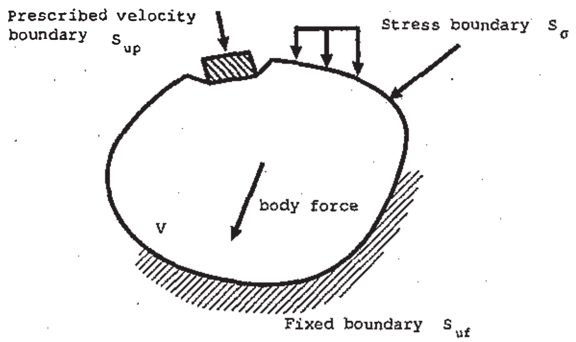

Figure 2. Targeted rigid-plastic boundary value problem.

Associated flow rule for the piecewise linear yielding locus is expressed as follows.

$$\underline{\dot q}^p = \underline{\underline{N}}\underline{\dot\lambda} \tag{10}$$

where $\underline{\dot\lambda}$ is a vector arranged non-negative plastic multipliers of each planes.

Compatibility condition of stresses and plastic strain rates can be expressed as

$$\left(\underline{h} - \underline{\underline{N}}^T \underline{Q}\right) \cdot \underline{\dot\lambda} = 0. \tag{11}$$

Equation (11) insists the complementary condition between stresses and plastic strain rates. This inequality insists the following two possibilities.

- A stress state is on the yielding locus, i.e. $h_i - \underline{n}_i \cdot \underline{Q} = 0$, or
- A stress state is within the yielding locus and no plastic strain rates are occurred, i.e., $\dot\lambda_i = 0$.

Targeted boundary value problems and spatial discretization. Targeted rigid-plastic boundary value problems are schematically shown in figure 2. For the simplicity of the discussion, only fixed boundaries and free surface boundaries (no traction on stress boundaries) are considered.

$$\underline{\dot u} = \underline{0} \text{ on } S_{uf}, \quad \underline{t} = \underline{0} \text{ on } S_\sigma. \tag{12}$$

External loads are only body forces $m\rho\underline{b}$, where scalar m is a load factor.

Velocity field $\underline{\dot u}$ is spatially discretized by shape functions $\underline{\underline{S}}$ and nodal velocities $\underline{\dot u}_N$, as

$$\underline{\dot u} = \underline{\underline{S}}\underline{\dot u}_N. \tag{13}$$

For rigid-plastic materials, plastic strain rates $\underline{\dot q}^p$ are associated with nodal velocities $\underline{\dot u}$ and plastic multipliers $\underline{\dot\lambda}$ via associated flow rule in the following manner,

$$\underline{\dot q}^p = \underline{\underline{B}}\underline{\dot u}_N = \underline{\underline{N}}\underline{\dot\lambda}. \tag{14}$$

Then, let us consider a targeted boundary value problem in a weak form. According to virtual work rate principle, a following equation in weak form can be obtained for any arbitrary kinematically admissible velocity field $\underline{\dot u}$, its corresponding strain rate field $\underline{\dot q}^p$ and for any arbitrary stress field $\underline{Q}$.

$$\int_V \underline{Q} \cdot \underline{\dot q}^p dV = \int_V m\rho\underline{b} \cdot \underline{\dot u} dV \tag{15}$$

This weak form equation can be transformed into a following equation by introducing spatial discretization,

$$\underline{\dot u}_N \cdot \int_V \underline{\underline{B}}^T \underline{Q} dV = m\underline{\dot u}_N \cdot \int_V \rho\underline{\underline{S}}^T \underline{b} dV \tag{16}$$

For the concise notation, volume integration in equation (16) is omitted in this paper.

Velocities on the fixed velocity boundaries S_{uf} can be expressed as follows,

$$\underline{0} = \underline{\underline{D}}_f \underline{\dot u}_N, \tag{17}$$

where tensor $\underline{\underline{D}}_f$ has zero or unity components. Its component $[D_f]_{ij}$ is unity, if the i-th fixed velocity condition is corresponding to the j-th component of nodal velocity vector $\underline{\dot u}_N$, else, $[D_f]_{ij}$ is zero.

Finally, weak form equation can be expressed as follows,

$$\underline{\dot u}_N \cdot \underline{\underline{B}}^t \underline{Q} = m\underline{\dot u}_N \cdot \underline{f} + \underline{0} \cdot \underline{\underline{D}}_f^T \underline{g}_f. \tag{18}$$

where scalar m is a load factor.

The internal dissipation energy rate $\dot W_{int}$ for a trial kinematically admissible velocity field is expressed in a spatially discretized form as follows,

$$\begin{aligned} \dot W_{int} &= \int_V \underline{Q}^p \cdot \underline{\dot q}^p dV \\ &= \int_V \underline{\dot\lambda} \cdot \underline{\underline{N}}^T \underline{Q}^p dV = \int_V \underline{\dot\lambda} \cdot \underline{h} dV \end{aligned} \tag{19}$$

where $\underline{Q}^p$ are stresses corresponding to a given velocity field. Plastic multipliers vector $\underline{\dot\lambda}$ are evaluated at integral points in numerical calculations. According to Hill's maximum plastic work principle, a following inequality holds,

$$\int_V \underline{h} \cdot \underline{\dot\lambda} = \int_V \underline{Q}^p \cdot \underline{\dot q}^p dV \geq \int_V \underline{Q}^* \cdot \underline{\dot q}^p dV, \tag{20}$$

where $\underline{Q}^*$ is the exact stress solution. This inequality can be transformed as follows by spatial discretization,

$$\dot W_{int} = \underline{h} \cdot \underline{\dot\lambda} \geq +m\underline{\dot u}_N \cdot \underline{f} = \dot W_{ext} \tag{21}$$

Linear programming formulation. Let us consider upper bound analysis for this boundary value problem. After some calculation of equation (21), the upper bound of a loading factor m is evaluated as

$$m \leq \frac{\underline{h} \cdot \dot{\underline{\lambda}}}{(\dot{\underline{u}}_N \cdot \underline{f})} \rightarrow \min \tag{22}$$

By noting that the external plastic work rate $\dot{W}_{ext}$ and the internal dissipation energy rate $\dot{W}_{int}$ are homogeneous in the first order of velocities, a loading factor is independent of the amount of the velocities themselves. Therefore, the additional condition of $\dot{\underline{u}}_N \cdot \underline{f} = 1$ doesn't loose generality in the calculation of a loading factor.

Thus, LP formulation of the upper bound method is derived as follows,

$\underline{h} \cdot \dot{\underline{\lambda}} \rightarrow \min$ subject to

$$\dot{\underline{u}}_N \cdot \underline{f} = 1,\ \dot{\underline{\lambda}} \geq \underline{0},\ \underline{\underline{B}}\dot{\underline{u}}_N = \underline{\underline{N}}\dot{\underline{\lambda}},\ \underline{\underline{D}}_f \dot{\underline{u}}_N = \underline{0} \tag{23}$$

Moreover, as nodal velocity vector $\dot{\underline{u}}$ is formally divided into two non-negative variables; $\dot{\underline{u}}_N = \dot{\underline{u}}_N^+ - \dot{\underline{u}}_N^-$, where $\dot{\underline{u}}_N^+ \geq \underline{0}$ and $\dot{\underline{u}}_N^- \geq \underline{0}$, a standard form LP can be derived.

$$[\underline{h}^t, \underline{0}^t, \underline{0}^t] \begin{Bmatrix} \dot{\underline{\lambda}} \\ \dot{\underline{u}}_N^+ \\ \dot{\underline{u}}_N^- \end{Bmatrix} \rightarrow \min \qquad \text{subject to}$$

$$\begin{bmatrix} \underline{\underline{N}} & -\underline{\underline{B}} & +\underline{\underline{B}} \\ \underline{0}^t & +\underline{f}^t & -\underline{f}^t \\ \underline{\underline{0}} & +\underline{\underline{D}}_f & -\underline{\underline{D}}_f \end{bmatrix} \begin{Bmatrix} \dot{\underline{\lambda}} \\ \dot{\underline{u}}_N^+ \\ \dot{\underline{u}}_N^- \end{Bmatrix} = \begin{Bmatrix} \underline{0} \\ 1 \\ \underline{0} \end{Bmatrix}$$

$$\dot{\underline{\lambda}} \geq \underline{0},\ \dot{\underline{u}}_N^+ \geq \underline{0},\ \dot{\underline{u}}_N^- \geq \underline{0} \tag{24}$$

where vectors noted here mean column vectors, and superscript t means transpose of column vectors, i.e., row vectors.

2.3 *Extention to work hardening materials*

Extention of limit analysis to rigid–work hardening materials is discussed in this section. For the simplicity of the discussion, a linearly hardening model with piecewisely linearised yield functions is considered. According to the linearly hardening model, strengths $\underline{h}$ are linear to the magnitude of plastic deformation, i.e., a plastic multiplier λ (König 1976).

$$\underline{h} = \underline{\underline{H}}\underline{\lambda} + \underline{h_0} \tag{25}$$

where tensor $\underline{\underline{H}}$ is hardening coefficient tensor and vector $myvech_0$ is an initial strength. By noting that a plastic multiplier λ is a monotonous increasing function with time, a strength vector $\underline{h}$ at time τ can be obtained by the following integration.

$$\underline{h}(\tau) = \int_{t=0}^{\tau} \underline{\underline{H}}\dot{\underline{\lambda}} dt + \underline{h_0} = \underline{\underline{H}} \int_{t=0}^{\tau} \dot{\underline{\lambda}} dt + \underline{h_0} \tag{26}$$

Then, internal dissipation W_{int} done during $\tau = 0 \sim T$ is considered. It is assumed that kinematically admissible velocities $\dot{\underline{u}}$ are constant with time to avoid the time domain integration and save the amount of calculations. This procedure is different from the ordinary elasto-plastic analysis, where step by step integration is used in the time domain.

$$\begin{aligned} W_{int} &= \int_{\tau=0}^{\tau=T} \dot{W}_{int}(\tau) d\tau \\ &= \int_{\tau=0}^{\tau=T} \left[\dot{\underline{\lambda}} \cdot \underline{\underline{H}} \left(\int_{t=0}^{t=\tau} \dot{\underline{\lambda}} dt \right) + \dot{\underline{\lambda}} \cdot \underline{h_0} \right] d\tau \\ &= \frac{1}{2} T^2 \dot{\underline{\lambda}} \cdot \underline{\underline{H}}\dot{\underline{\lambda}} + T\dot{\underline{\lambda}} \cdot \underline{h_0} \end{aligned} \tag{27}$$

The same assumption is applied for the calculation of external plastic work W_{ext}. Consequently,

$$W_{ext} = \int_{\tau=0}^{\tau=T} \dot{W}_{ext} d\tau = mT\dot{\underline{u}}_N \cdot \underline{f} \tag{28}$$

Finally, the upper bound theorem is integrated by time to estimate the upper bound value of a load factor m,

$$m \leq \frac{\frac{1}{2} T \dot{\underline{\lambda}} \cdot \underline{\underline{H}}\dot{\underline{\lambda}} + \dot{\underline{\lambda}} \cdot \underline{h_0}}{\dot{\underline{u}}_N \cdot \underline{f}} \rightarrow \min \tag{29}$$

Additionally, it is convenient to normalise the external plastic work $W_{ext} = 1$ in the calculation, as same as the ordinal upper bound analysis. Moreover, if a hardening coefficient tensor $\underline{\underline{H}}$ is positive definite, equation (29) is concluded to the quadratic programming.

2.4 *Proposed calculation procedure*

No limit of plastic deformation for the linearly hardening model is given in the previous section. Accordingly, a strength can be increased infinitely. This is far from the real behaviour of materials. Therefore, selection of a time parameter T is important to evaluate the stability rationally.

Physical meaning of a time parameter T is explained here. Though parameter T holds a time dimension, it is not necessarily a physical time itself. However, a product of T and a corresponding velocity field must be a displacement. It should be reminded that absolute values of a velocity field are meaningless due to the first order homogeneity of velocities in both the internal dissipation rate $\dot{W}_{int}$ and the external plastic work rate $\dot{W}_{ext}$. Similarly, the absolute value of time parameter T is meaningless. It should be emphasised again that a product of a time parameter T and its corresponding velocity field must be a physical displacement.

As pointed out previously, the additional stability criteria of a deformation, which ensures that no work-softening behaviour is occurred in anywhere of a domain, is applied to estimate the stability of soil structure—reinforcement members system, as follows.

1 Assume initial time parameter T_i.

2 Evaluate the optimum velocity field based on the quadratic programming equation (29).

3 Calculate strains and displacements in all the region by the product of the obtained velocity field and the assumed time parameter.

4 Check the strains and displacements in all the region to ensure that they are within threshold values.

   a) If strains or displacements are more than the threshold values, modify a time parameter $T_{i+1} = T_i - \delta T$ to return to step 1.

   b) If strains and displacements are all with the threshold values, modify a time parameter $T_{i+1} = T_i - \delta T$ and check the convergence of the calculation.

      * A load factor m is converged. Stop the calculation.
      * A load factor m is not increased, Return to step 1.

It should be noted that strains and displacements are checked after the optimisation of the quadratic programming. Only kinematically and plastically admissible velocity fields are required for the upper bound analysis. If additional restrictions of the deformation were applied in the optimisation process, its solution might be overestimated.

2.5 *Practical interpretation of the proposed method*

Practical interpretation of the proposed method is discussed in this section.

The proposed method does not require a large amount of calculations in comparison with an evolutional (step by step) method. This is a big advantage for numerical calculations. An assumed velocity field is constant with time to avoid the time integral calculations. This assumption might be applicable, if a collapse mechanism is not changed drastically with time. If the assumption of the positive definiteness of hardening coefficient tensor $\underline{\underline{H}}$ is applicable, numerical calculations are stable due to the convexity of the analysis.

A load factor, a time parameter and its corresponding optimised velocity field are obtained by this method. A distribution of displacements can be estimated from these results. A distribution of displacements is good information on the practical management, as displacements can be measured easily in the practical sites. Member forces of reinforcements can be also estimated. A distribution of these member forces is good information to consider the locations of the reinforcement members.

The proposed method is also applicable to the elasto-plastic problems, if only loading processes are considered. Elasto-plastic problems under monotonic loading conditions are essentially same to the rigid-plastic problems with zero initial strengths. It is observed in some practical cases that reinforcement members are still under the elastic state, although surrounding soils are in the plastic state. However, the proposed method is applicable to this situation.

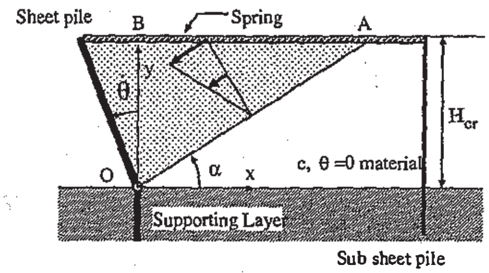

Figure 3. Example boundary value problem.

3 NUMERICAL EXAMPLE

A very simple example is considered to demonstrate the proposed method. The stability problem of a self-standing sheet-pile wall is considered. A sheet pile with plastic bending moment M_y is installed sufficiently to the supporting layer. Soils behind the wall is c, $\phi = 0$ material with a density ρ. An elastic spring with the elastic modulus k is pin-connected to the top of the sheet pile. The other end of the spring is firmly supported and its influences are negligible. Frictional effects between soils, a sheet-pile and a spring are also negligible.

It should be calculated for all the kinematically admissible velocity field to obtained the best optimised solution. However, only one deformation mode shown in figure 3 is considered here for the demonstration.

A velocity field of the soil for this mode is

$$v_x = -\dot{\theta}(y - \tan\alpha \cdot x),$$

$$v_y = -\dot{\theta}\tan\alpha \cdot (y - \tan\alpha \cdot x) \qquad (30)$$

Then, strain rates are calculated as

$$\dot{\varepsilon}_x = -\dot{\theta}\tan\alpha,\ \dot{\varepsilon}_y = \dot{\theta}\tan\alpha,$$

$$\dot{\gamma} = 2\dot{\varepsilon}_{xy} = -\dot{\theta}(1 + \tan^2\alpha) \qquad (31)$$

The internal dissipation is occurred at the soils, the sheet pile and the spring. On the other hand, the external plastic work is done by the self weight of the soils.

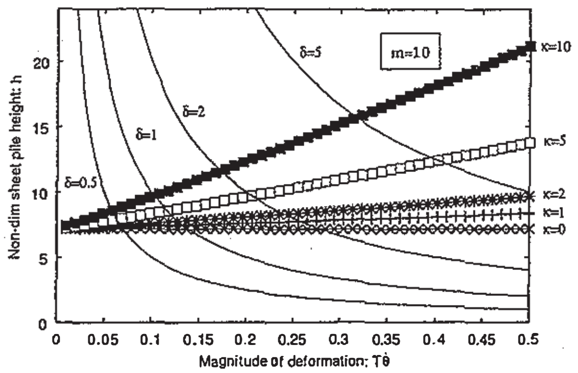

Figure 4. Numerical result.

By optimising about the parameter α which governs the area of plastic deformation, the stational point is at the angle of $\tan\alpha = 1$. Therefore, non-dimensional critical height of the sheet pile is a solution of the following cubic equation.

$$\frac{1}{6}h^3 - \left(1 + \frac{1}{2}\kappa T\dot{\theta}\right)h^2 - m = 0 \tag{32}$$

where $l = c/(\rho g)$ is a specific length, $h = H_{cr}/l$ is a non-dimensional sheet pile height, $m = M_y/(cl^2)$ is a non-dimensional plastic bending moment of a sheet pile and $\kappa = k/c$ is a representative elastic modulus of a spring.

A numerical result in case of a non-dimensional plastic bending moment $m = 10$ is shown in figure 4. The horizontal axis means the magnitude of plastic deformation $T\theta$. This value coincides with the magnitude of plastic (engineering) shear strain of the soil behind the wall. Parameter δ shown as thin lines in figure 4 is non-dimensional horizontal displacement at the top of the sheet pile normalised by a specific length l. It is clearly observed that non-dimensional height $\sim$ magnitude of deformation relations depend on the elastic moduli of springs. If additional conditions for the threshold strains or displacements are given, the critical height of the sheet pile for this mode can be evaluated from figure 4.

4 CONCLUSIONS

Theoretical methodology to evaluate the safety of soil structures with reinforcement members from the point of limit analysis is discussed in this paper. The conclusions of this paper is summarised as follows.

- The extention of the upper bound analysis to rigid-work hardening materials is presented. The formulation is concluded to the quadratic programming.
- The evaluation method of the stability of soil structures and reinforcement members system is proposed based on the quadratic programming and the additional deformation conditions of the threshold values for strains and displacements.
- The proposed method does not require a large amount of calculations in comparison with an evolutional (step by step) method.
- A load factor, a time parameter and its corresponding optimised velocity field are obtained by this method.
- The proposed method is also applicable to the elasto-plastic problems, if only loading processes are considered.

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